

RATE OF CONVERGENCE OF DIFFERENCE APPROXIMATIONS FOR UNIFORMLY NONDEGENERATE ELLIPTIC BELLMAN'S EQUATIONS

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ABSTRACT. We show that the rate of convergence of solutions of finite-difference approximations for uniformly elliptic Bellman's equations is of order at least $h^{2/3}$, where h is the mesh size. The equations are considered in smooth bounded domains.

The convergence of and error estimates for monotone and consistent approximations to fully nonlinear, first-order PDEs were established a while ago by Crandall and Lions [5] and Souganidis [24].

The convergence of monotone and consistent approximations for fully nonlinear, possibly degenerate second-order PDEs was first proved in Barles and Souganidis [4]. In a series of papers Kuo and Trudinger [20, 21, 22] also looked in great detail at the issues of regularity and existence of such approximations for uniformly elliptic equations.

There is also a probability part of the story, which started long before see Kushner [18], Kushner and Dupuis [19], also see Pragarauskas [23].

However, in the above cited articles apart from [5, 24], related to the first-order equations, no rate of convergence was established. One can read more about the past development of the subject in Barles and Jakobsen [3] and the joint article of Hongjie Dong and the author [7]. We are going to discuss only some results concerning *second-order* Bellman's equations, which arise in many areas of mathematics such as control theory, differential geometry, and mathematical finance (see Fleming and Soner [8], Krylov [9]) and which are most relevant to the results of the present article.

The first estimates of the rate of convergence for second-order degenerate Bellman's equations appeared in 1997 (see [11]). For equations with constant "coefficients" and arbitrary monotone finite-difference approximations it was proved in [11] that the rate of convergence is $h^{1/3}$ if the error in approximating the true operators with finite-difference ones is of order h on *three* times continuously differentiable functions. The order becomes $h^{1/2}$ if the error in approximating the true operators with finite-difference ones is of order h^2 on *four* times continuously differentiable functions (see

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Remark 1.4 in [11], which however contains an arithmetical error albeit easily correctable. Also see Theorem 5.1 in [11]). The main idea of [11] that the equation and its finite-difference approximation should play symmetric roles is also used in the present article. The proofs in [11] are purely analytical (in contrast with what one can read in some papers mentioning [11]) even though sometimes probabilistic *interpretation* of some statements are also given. The next step was done in [12] where the so-called method of “shaking the coefficients” was introduced to deal with the case of *degenerate* parabolic Bellman’s equations with variable coefficients. The two sided error estimates were given: from the one side of order $h^{1/21}$ and from the other $h^{1/3}$. Here h (unlike in [11]) was naturally interpreted as the mesh size and the approximating operators were assumed to approximate the true operator with error of order h on *three* times continuously differentiable functions. Until now it is not known whether or not it is possible to improve $1/21$ in the general setting of [12].

However, what is possible is that one can get better estimates if one uses some special approximations, say providing the error of order h^2 of approximating the main part of the true operator on *four* times continuously differentiable functions. This was already mentioned in Remark 1.4 of [11] and used by Barles and Jakobsen in [1] to extend the results in [11] to equations with variable lower-order “coefficients”.

One can also consider special finite-difference approximations, for instance, only containing pure second-order differences in place of second-order derivatives, when this h^2 approximating error is automatic. In such cases the optimal rate $h^{1/2}$ was obtained in the joint work of Hongjie Dong and the author [7] for parabolic Bellman’s equations with Lipschitz coefficients in domains. Both ideas of symmetry and “shaking the coefficients” is used in [7] as well as in [6]. In the paper by Hongjie Dong and the author [6] we consider among other things weakly nondegenerate Bellman’s equations with *constant* “coefficients” in the whole space and obtain the rate of convergence h , where h is the mesh size. It may be tempting to say that this result is an improvement of earlier results, however it is just a better rate under different conditions.

It is worth noting that the set of equations satisfying the conditions in [7] is smaller than the one in the papers by Barles and Jakobsen [2, 3], the results of which obtained by using the theory of viscosity solutions guarantee the rate $h^{1/5}$. However, in the examples given in [2, 3] of applications of the general scheme, for us to get the rate $h^{1/2}$, we (only) need to add the requirement that the coefficients be twice differentiable (see [14]) and in [2, 3] they are only assumed to be once differentiable and still the rate $h^{1/5}$ is guaranteed. One more point to be noted is that in [3] parabolic equations are considered with various types of approximation such as Crank-Nicholson and splitting-up schemes related to the time derivative.

In the present paper we add two more restrictions on the equations from [7]: a) we require the coefficients to be in $C^{1,1}(\mathbb{R}^d)$, b) we require the equation to be uniformly nondegenerate. In this case we obtain the rate of convergence $h^{2/3}$, which was announced previously in [13] for equations in the whole space. This time “shaking” is not needed as we explain in Remark 5.1. It would be very interesting to find a way to derive the results of the present paper by using methods from [1, 2, 3] or other methods based on the theory of viscosity solutions.

1. MAIN RESULTS

Let A be a set and let

$$a_k^\alpha = a_k^\alpha(x), \quad b_k^\alpha = b_k^\alpha(x), \quad c^\alpha = c^\alpha(x), \quad f^\alpha(x)$$

be real-valued functions of (α, x) defined on $A \times \mathbb{R}^d$ for $k = \pm 1, \dots, \pm d_1$. We assume that, for each $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, these functions are bounded with respect to $\alpha \in A$. Also let some vectors

$$l_k = (l_{1k}, \dots, l_{dk}) \in B := \{x : |x| \leq 1\} \subset \mathbb{R}^d$$

be defined for $k = \pm 1, \dots, \pm d_1$. This somewhat unusual range of k turns out to be convenient as we explain in Remark 1.1. Consider the following Bellman’s equation arising, for instance, in the theory of controlled diffusion processes (see, for instance, [8], [9]):

$$\sup_{\alpha \in A} [L^\alpha v(x) - c^\alpha(x)v(x) + f^\alpha(x)] = 0, \quad (1.1)$$

where

$$L^\alpha v(x) = \sum_{i,j=1}^d a_{,ij}^\alpha(x) D_{ij} v(x) + \sum_{i=1}^d b_{,i}^\alpha(x) D_i v(x), \quad (1.2)$$

$$D_i = \frac{\partial}{\partial x_j}, \quad D_{ij} = D_i D_j,$$

$$a_{,ij}^\alpha(x) = \sum_{|k|=1}^{d_1} a_k^\alpha(x) l_{ik} l_{jk}, \quad b_{,i}^\alpha(x) = \sum_{|k|=1}^{d_1} b_k^\alpha(x) l_{ik}. \quad (1.3)$$

As follows from the title we will be dealing with uniformly elliptic operators and it is well-known that for any uniformly elliptic operator of type (1.2) there exist constant vectors l_k (independent of α) such that representation (1.3) holds. In addition, the regularity properties of $a_{,ij}^\alpha$ are inherited by a_k^α (see, for instance, Theorem 1.1 below). This is also known for many degenerate elliptic operators (see, for instance, [14]).

In what follows we adopt the summation convention over all “reasonable” values of repeated indices. Observe that

$$a_{,ij}^\alpha D_{ij} v = a_k^\alpha D_{l_k}^2 v, \quad b_{,i}^\alpha D_i v = b_k^\alpha D_{l_k} v,$$

where $D_{l_k}^2 v$ and $D_{l_k} v$ are the second and the first derivatives of v in the direction of l_k , that is

$$D_{l_k}^2 v = l_{ik} l_{jk} D_{ij} v, \quad D_{l_k} v = l_{ik} D_i v.$$

Therefore equation (1.1) is rewritten as

$$\sup_{\alpha \in A} [a_k^\alpha(x) D_{l_k}^2 v(x) + b_k^\alpha(x) D_{l_k} v(x) - c^\alpha(x) v(x) + f^\alpha(x)] = 0, \quad (1.4)$$

where, naturally, the summations with respect to k are performed inside the sup sign. We approximate solutions of (1.4) by solutions of finite-difference equations obtained after replacing $D_{l_k}^2 v$ and $D_{l_k} v$ with second- and first-order differences, respectively, taken in the direction of l_k .

For any $x, \xi \in \mathbb{R}^d$, $h > 0$, and function ϕ on \mathbb{R}^d introduce

$$T_{h,\xi} \phi(x) = \phi(x + h\xi), \quad \delta_{h,\xi} = \frac{T_{h,\xi} - 1}{h}, \quad \Delta_{h,\xi} = \frac{T_{h,\xi} - 2 + T_{h,-\xi}}{h^2}.$$

When ξ is one of the l_k 's we use the notation

$$\delta_{h,k} = \delta_{h,l_k}, \quad \Delta_{h,k} = \Delta_{h,l_k}, \quad k = \pm 1, \dots, \pm d_1,$$

in which the finite difference approximation of (1.4) is the following

$$\sup_{\alpha \in A} [a_k^\alpha(x) \Delta_{h,k} v(x) + b_k^\alpha(x) \delta_{h,k} v(x) - c^\alpha(x) v(x) + f^\alpha(x)] = 0. \quad (1.5)$$

Assumption 1.1. We are given a function g on \mathbb{R}^d and two constants $\delta, K \in (0, \infty)$ such that for all $\alpha \in A$ and k on \mathbb{R}^d we have

$$a_k^\alpha \geq \delta, \quad c^\alpha \geq \delta$$

and for $\phi = g, a_j^\alpha, b_k^\alpha, c^\alpha, f^\alpha$, $\alpha \in A$, and $j, k \in \{\pm 1, \dots, \pm d_1\}$ we have that $\phi \in C^{1,1}(\mathbb{R}^d)$ and

$$\|\phi\|_{C^{1,1}(\mathbb{R}^d)} \leq K.$$

Assumption 1.2. (i) We have $l_k = -l_{-k}$, $a_k^\alpha = a_{-k}^\alpha$, $k = \pm 1, \dots, \pm d_1$.

(ii) There exists an integer $1 \leq d_0 \leq d_1$ such that for

$$\Lambda := \{l_k, k = \pm 1, \dots, \pm d_1\}, \quad \mathcal{L} := \{l_{\pm 1}, \dots, l_{\pm d_0}\}$$

we have $0 \in \mathcal{L}$ and

$$\mathcal{L} + \mathcal{L} \supset \Lambda \supset \{l' + l'' : l', l'' \in \mathcal{L}, l' \neq l''\}.$$

(iii) The coordinates of l_k are rational numbers and $\text{Span } \Lambda = \mathbb{R}^d$.

To justify Assumptions 1.1 and 1.2 we remind the reader Theorem 3.1 of [15] in which $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and \mathcal{S}_{δ_1} , $\delta_1 > 0$, is the set of symmetric $d \times d$ -matrices a such that for any $\xi \in \mathbb{R}^d$

$$\delta_1 |\xi|^2 \leq \langle a\xi, \xi \rangle \leq \delta_1^{-1} |\xi|^2$$

($\langle \cdot, \cdot \rangle$ stands for the scalar product).

Theorem 1.1. *There exists a set $\{l_1, \dots, l_n\} \subset \mathbb{Z}^d$ such that for any its extension $\{l_1, \dots, l_m\}$, $m \geq n$, there exist real-analytic functions $\lambda_1(a), \dots, \lambda_m(a)$ on \mathcal{S}_{δ_1} possessing there the following properties:*

$$a \equiv \sum_{k=1}^m \lambda_k(a) l_k l_k^*, \quad \lambda_k(a) \geq \delta, \quad \forall k,$$

where the constant $\delta > 0$.

Remark 1.1. Theorem 1.1 implies that any uniformly nondegenerate equation of type (1.1) can be written as (1.4) with the coefficients satisfying Assumption 1.1 as long as the coefficients in (1.1) are bounded and twice continuously differentiable with C^2 -norm controlled by a constant independent of α and c is uniformly bounded away from zero. If we take $\{l_1, \dots, l_n\}$ from Theorem 1.1 and define $\mathcal{L} = \{0, \pm l_1, \dots, \pm l_n\}$ and

$$\Lambda = \{l' + l'' : l', l'' \in \mathcal{L}, l' \neq l''\},$$

then Assumption 1.2(ii) will be obviously satisfied and owing to Theorem 1.1 Assumption 1.1 will still be preserved. Assumption 1.2(i) is of a technical nature and easily satisfied just by redefining a_k^α if necessary which is possible since the above Λ is symmetric and $D_l^2 = D_{-l}^2$. One could exclude Assumption 1.2(i) on the expense of more complicated formulation of Assumption 1.2(ii), which, by the way, is needed in order to apply the results of [17] about interior estimates of second-order differences of approximate solutions. It is also worth saying that by assumption our l_k have lengths ≤ 1 . Therefore one should normalize those l_k 's from Theorem 1.1, which are different from zero, absorbing the extra factors in $\lambda_k(a)$.

Also observe that Assumption 1.2 requires that $l_k = 0$ for some k . Including the zero vector in Λ turns out to be convenient from technical point of view. The coefficient a_k^α corresponding to this vector can be set to equal, say 1, because the corresponding finite-difference operator is just zero.

Finally, talking about our assumptions we point out that Assumptions 1.2(ii)(iii) are used in Section 3 when we apply some results of [16] and [17] to finite difference equations in domains.

We suppose that we are given a $\psi \in C^2(\mathbb{R}^d)$ such that

$$\Omega = \{x : \psi(x) > 0\}$$

is a bounded domain and $|D\psi| \geq 1$ on $\partial\Omega$. Equation (1.4) is considered in Ω with v subject to the boundary condition $v = g$ on $\partial\Omega$.

Introduce

$$\Omega_h = \{x \in \Omega : x + hB \subset \Omega\}, \quad \partial_h \Omega = \mathbb{R}^d \setminus \Omega_h.$$

Observe that $\partial_h \Omega$ contains points which are very far from Ω . This turns out to be convenient in our constructions.

Here are our main results in which the above assumptions are supposed to hold and

$$\rho(x) = \text{dist}(x, \Omega^c), \quad \text{where} \quad \Omega^c = \mathbb{R}^d \setminus \Omega.$$

Naturally, $\rho = 0$ on Ω^c .

Theorem 1.2. *There are constants $N \in [0, \infty)$ and $h_0 > 0$ such that for all $h \in (0, h_0]$ the following is true:*

(i) *Equation (1.5) in Ω_h with boundary condition $v = g$ on $\partial_h \Omega$ has a unique bounded Borel solution v_h .*

(ii) *On \mathbb{R}^d*

$$\rho^{-1}|v_h - g|, \quad |\delta_{h,i}v_h|, \quad (\rho - 6h)|\delta_{h,i}\delta_{h,j}v_h| \leq N \quad (1.6)$$

for any i, j and for any $x, y \in \mathbb{R}^d$

$$|v_h(x) - v_h(y)| \leq N(|x - y| + h). \quad (1.7)$$

Theorem 1.3. *There exists a unique $v \in C_{loc}^{1,1}(\Omega) \cap C^{0,1}(\mathbb{R}^d)$ satisfying equation (1.4) in Ω (a.e.) and equal g on Ω^c . Furthermore, $\rho|D^2v| \leq N$ in Ω (a.e.), where N is a constant.*

This theorem is proved in exactly the same way in which Theorem 8.7 of [15] is proved. On this way one uses Theorem 1.2, the fact that the derivatives of v are weak limits of finite differences of v_h as $h \downarrow 0$ (see the proof of Theorem 8.7 of [15]), and the fact that there are sufficiently many second order derivatives in directions of l_i, l_j to conclude from their boundedness that the Hessian of v is bounded.

Remark 1.2. In the theory of fully nonlinear elliptic equations much more general results than Theorem 1.3 under much weaker conditions are known. For instance, it is proved in [25] that if $\beta \in (0, 1)$ is sufficiently small and $a^\alpha, b^\alpha, c^\alpha, f^\alpha \in C^\beta(\bar{\Omega})$ with $C^\beta(\bar{\Omega})$ -norms bounded by a constant independent of α , then $v \in C_{loc}^{2+\beta}(Q) \cap C^{0,1}(\bar{Q})$.

The results in [25] and other classical texts on the theory of fully nonlinear elliptic equations are obtained on the basis of very deep facts, using very sophisticated and beautiful techniques, and require a series of long arguments in the end of which the reader learns a lot of various facts from the theory of PDEs and functional analysis. In contrast, our Theorem 1.3 is obtained on the sole basis of the discrete maximum principle combined with elementary albeit quite long computations (see [16] and [17]) involving discrete versions of Bernstein's method.

Theorem 1.4. *There are constants $N \in [0, \infty), h_0 > 0$ such that for all $h \in (0, h_0]$ we have $|v_h - v| \leq Nh^{2/3}$ on \mathbb{R}^d .*

Example 1.1. Consider the following uniformly nondegenerate analog of the Monge-Ampère equation

$$\det(-D_{ij}v - \gamma^2 \delta^{ij} \Delta v) = (f_+)^d, \quad (1.8)$$

where $\gamma > 0$ is a parameter and $f \in C^{1,1}(\mathbb{R}^d)$. It is well known (see, for instance, [10]) that equation (1.8) supplied with the requirement that the

matrix $(D_{ij}v + \gamma^2 \delta^{ij} \Delta v)$ be negative definite is equivalent to the following single Bellman's equation:

$$\sup_{\substack{a=a^* \geq 0, \\ \text{trace } a=1}} [(a_{ij} + \gamma^2 \delta^{ij}) D_{ij}v + (\det^{1/d} a) f] = 0. \quad (1.9)$$

The above theory is applicable to (1.9) and shows that we can approximate the solutions of (1.8) with a $C^{1,1}$ boundary condition satisfying $(D_{ij}v + \gamma^2 \delta^{ij} \Delta v) \leq 0$ with solutions of corresponding finite-difference equations. It is to be noted however that the number d_1 related to the number of directions needed to write (1.9) in the form (1.4) will depend on γ and will go to infinity as $\gamma \rightarrow 0$.

Also observe that one can prove that if Ω is a strictly convex domain, and v^γ are solutions of (1.9) with zero boundary condition, then $|v^\gamma - v^0| \leq N\gamma$, where v^0 is a (generalized or viscosity) solution of (1.8) with $\gamma = 0$.

The rest of the article is organized as follows. In Section 2 we prove (1.7) assuming (1.6) for equations more general than Bellman's equations. Section 3 contains two results from [16] and [17] needed to prove (1.6) in Section 4. In the final rather long Section 5 we present the proof of Theorem 1.4.

We use N to denote various constants which may change from one appearance to another. Sometimes we specify what they are depending on or independent of. However sometimes we do not do that. In these situations it is understood that they are independent of anything which is allowed to change like x, h, ε, \dots

2. ON THE LIPSCHITZ CONTINUITY OF v_h

The setting and notation in this section are different from the ones in Section 1. Let

$$\Lambda := \{\ell_k; k = 1, \dots, d_1\}$$

be a symmetric subset of $B = \{x : |x| < 1\}$.

Let $H(p, x, u)$ be a real-valued function given for

$$p \in \mathbb{R}, \quad x \in \mathbb{R}^d, \quad u = (u', u''), \quad u' = (u'_0, u'_1, \dots, u'_{d_1}), \quad u'' = (u''_1, \dots, u''_{d_1}).$$

Fix two constants $\delta \in (0, 1], K \in [0, \infty)$.

Assumption 2.1. The function $H(p, x, u)$ is locally Lipschitz continuous with respect to (p, u) . Furthermore, at all points of differentiability of H with respect to (p, u) we have

$$\delta \leq H_{u''_j} \leq K, \quad j = 1, \dots, d_1, \quad H_{u'_0} \leq -\delta,$$

$$|H_{u'_j}| \leq K, \quad j = 0, \dots, d_1, \quad |H_p(p, x, u)| \leq K(1 + |u|).$$

Introduce

$$H(x, u) = H(0, x, u).$$

The following lemma is an easy consequence of the mean value theorem in the form of Hadamard.

Lemma 2.1. *For any $p \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $u, v \in \mathbb{R}^{2d_1+1}$ there exist numbers a_1, \dots, a_{d_1} , b_1, \dots, b_{d_1} , c , and f such that*

$$\begin{aligned} H(p, x, u) - H(x, v) &= H(p, x, u) - H(p, x, v) + f \\ &= a_i(u''_i - v''_i) + b_i(u'_i - v'_i) + c(u'_0 - v'_0) + f, \\ \delta \leq a_i \leq K, \quad -K \leq c \leq -\delta, \quad |b_i| \leq K, \quad |f| \leq K|p|(1 + |v|) \end{aligned} \quad (2.1)$$

for all $i \in \{1, \dots, d_1\}$.

For any function v and $h > 0$ define

$$H_h[v](x) = H(x, v(x), \delta_h v(x), \Delta_h v(x)),$$

where

$$\begin{aligned} \delta_h v &= (\delta_{h,1} v, \dots, \delta_{h,d_1} v), \quad \Delta_h v = (\Delta_{h,1} v, \dots, \Delta_{h,d_1} v), \\ \delta_{h,k} &= \delta_{h,\ell_k}, \quad \Delta_{h,k} = \Delta_{h,\ell_k}. \end{aligned}$$

Theorem 2.2. *Let D be a domain in \mathbb{R}^d and let $h \in (0, \delta/K]$. Then there exists a constant N_0 depending only on δ, K , and d_1 such that for any two functions v' and v'' given on \mathbb{R}^d we have that in D_h*

$$N_0(v' - v'') + h^2(H_h[v'] - H_h[v'']) \leq (N_0 - \delta h^2) \sup_D (v' - v'')_+, \quad (2.2)$$

$$\sup_{\mathbb{R}^d} (v' - v'')_+ \leq \delta^{-1} \sup_{D_h} (H_h[v''] - H_h[v'])_+ + \sup_{\partial_h D} (v' - v'')_+. \quad (2.3)$$

Furthermore, if $H(x, 0)$ is bounded and we are given a bounded function g on \mathbb{R}^d , then the equation

$$H_h[v] = 0 \quad (2.4)$$

in D_h with boundary condition $v = g$ on $\partial_h D$ (in case $\partial_h D \neq \emptyset$) has a unique bounded solution. This solution is Borel measurable in x if g and $H(x, u)$ are Borel for each u .

Proof. As is usual for monotone finite-difference equations in the proof of solvability of (2.4) we rely on the method called Jacobi iteration in [21] and the Banach fixed point theorem in the space \mathfrak{B} of bounded functions on \mathbb{R}^d provided with the sup norm. We will follow an argument from [15] where the situation is somewhat different.

First we deal with (2.2). By Lemma 2.1, for $w = v' - v''$ we have on \mathbb{R}^d that

$$\begin{aligned} h^2(H_h[v'](x) - H_h[v''](x)) &= (ch^2 - h \sum_{j=1}^{d_1} b_j - 2 \sum_{j=1}^{d_1} a_j)w(x) \\ &\quad + hb_j w(x + h\ell_j) + a_j[w(x + h\ell_j) + v(x - h\ell_j)]. \end{aligned}$$

Since $a_j, b_j, -c \leq K$, and $h \in (0, \delta/K]$, there is a constant N_0 depending only on δ, K , and d_1 such that

$$N_0 + ch^2 - h \sum_{j=1}^{d_1} b_j - 2 \sum_{j=1}^{d_1} a_j \geq 0.$$

Furthermore,

$$hb_j w(x+h\ell_j) + a_j[w(x+h\ell_j) + w(x-h\ell_j)] = (hb_j + a_j)w(x+h\ell_j) + a_j w(x-h\ell_j)$$

and all the coefficients on the right are nonnegative in light of the fact that $h \in (0, \delta/K]$ and $a_j \geq \delta$. It follows that in D_h the left-hand side of (2.2) is less than

$$\begin{aligned} & (N_0 + ch^2 - h \sum_{j=1}^{d_1} b_j - 2 \sum_{j=1}^{d_1} a_j) \sup_D w_+ \\ & + \sum_{j=1}^{d_1} (hb_j + a_j) \sup_D w_+ + \sum_{j=1}^{d_1} a_j \sup_D w_+ \\ & = (N_0 + ch^2) \sup_D (v' - v'')_+ \leq (N_0 - \delta h^2) \sup_D (v' - v'')_+, \end{aligned}$$

which proves (2.2).

While proving (2.3) we may assume that

$$\sup_{\mathbb{R}^d} (v' - v'')_+ > \sup_{\partial_h D} (v' - v'')_+.$$

In that case

$$\sup_{\mathbb{R}^d} (v' - v'')_+ = \sup_{D_h} (v' - v'')_+$$

and (2.2) implies

$$N_0 (v' - v'')_+ \leq (N_0 - \delta h^2) \sup_{D_h} (v' - v'')_+ + h^2 (H_h[v''] - H_h[v'])_+.$$

After that it only remains to take the sups of both sides over D_h and collect like terms.

Now we prove the second assertion of the theorem. Observe that, as is easy to see, equation (2.4) in D_h with the boundary condition $v = g$ on $\partial_h D$ is equivalent to the following single equation:

$$v(x) = [N_0^{-1} h^2 H_h[v](x) + v(x)] I_{D_h}(x) + g(x) I_{\partial_h D}(x). \quad (2.5)$$

Then introduce an operator $T_h : v \rightarrow T_h v$ by

$$T_h v(x) := [N_0^{-1} h^2 H_h[v](x) + v(x)] I_{D_h}(x) + g(x) I_{\partial_h D}(x).$$

Since $H[0]$ is bounded by assumption, T_0 is bounded. By (2.2) we have that $T_h v = T_h v - T_h 0 + T_h 0$ is bounded if v is bounded, so that T_h is an operator in \mathfrak{B} , and moreover T_h is a contraction operator in \mathfrak{B} with contraction constant less than $1 - N_0^{-1} \delta h^2 < 1$. By the Banach fixed point theorem, equation (2.5) has a unique bounded solution. This solution can be obtained as the limit of $(T_h)^n 0$ as $n \rightarrow \infty$. Furthermore, T_h maps Borel functions into Borel

measurable ones if $H(x, u)$ is Borel with respect to x . Therefore, under this condition, given that g is Borel, all functions $T_h^n 0$ are Borel measurable, so that the solution v is also Borel measurable. The theorem is proved.

By taking first $v' = v$, $v'' = 0$ and then $v' = 0$, $v'' = v$ we obtain from (2.3) the following.

Corollary 2.3. *If v is a solution of (2.4) with boundary condition $v = g$ on $\partial_h D$, then*

$$\sup_{\mathbb{R}^d} v_+ \leq \delta^{-1} \sup_{D_h} (H_h[0])_+ + \sup_{\partial_h D} g_+, \quad \sup_{\mathbb{R}^d} v_- \leq \delta^{-1} \sup_{D_h} (H_h[0])_- + \sup_{\partial_h D} g_-.$$

We improve these estimates in Lemma 2.5 for more restricted range of h . We need a version of Lemma 8.5 of [15]. Let Ω be the set introduced in Section 1.

Lemma 2.4. *Assume that $\text{Span}(\Lambda) = \mathbb{R}^d$. Then there exist $h_0 \in (0, \infty)$ and a nonnegative function $\Psi \in C^2(\bar{\Omega})$ such that Ψ/ρ and ρ/Ψ are bounded in Ω , for $h \in (0, h_0]$ on Ω_h we have*

$$a_j \Delta_{h,j} \Psi + K \sum_{j=1}^m |\delta_{h,j} \Psi| \leq -1, \quad (2.6)$$

and such that there exist constants $M, \mu \geq 1$ for which the function $\Phi := M\Psi \ln(M/\Psi)$ satisfies

$$a_j \Delta_{h,j} \Phi + K \sum_{j=1}^m |\delta_{h,j} \Phi| \leq -\rho^{-1} \quad (2.7)$$

in $\Omega_{\mu h}$, whenever $\delta \leq a_j \leq K$.

Below by h_0 we mean a constant in $(0, \delta/K]$ for which the statement of Lemma 2.4 is true.

Lemma 2.5. *Assume that $\text{Span}(\Lambda) = \mathbb{R}^d$. Let $D \subset \Omega$, $h \in (0, h_0]$ and let v satisfy (2.4) in D_h and $v = g$ in $\partial_h D$, where $g \in C^{1,1}(\mathbb{R}^d)$. Then $|v - g| \leq N\rho$ on \mathbb{R}^d , where the constant N is independent of h .*

Proof. It suffices to prove that $|v - g| \leq N\Psi$ on \mathbb{R}^d if we continue Ψ outside Ω as zero. In order to do this we observe that by Lemma 2.1 with $p = 0$ at each point of D_h

$$H_h[g + N\Psi] = H_h[g + N\Psi] - H_h[0] + H_h[0]$$

$$= N(a_j \Delta_{h,i} \Psi + b_j \delta_{h,j} \Psi + c\Psi) + a_j \Delta_{h,i} g + b_j \delta_{h,j} g + cg + H(\cdot, 0),$$

where a_j, b_j, c are some numbers satisfying (2.1). Owing to (2.6) the last expression is negative if we take N large enough. This implies that $v - g \leq N\Psi$ by the maximum principle (that is by (2.3)) and by the fact that $v = g$ and $\Psi \geq 0$ in $\partial_h D$. The estimate $v - g \geq -N\Psi$ is proved similarly. The lemma is proved.

In the rest of the section we assume that

$$\text{Span}(\Lambda) = \mathbb{R}^d,$$

fix a constant $\mu \geq 1$, and take a constant $\nu \in [0, \infty)$. Define

$$\kappa = \mu \vee (2\nu)$$

and let D be an open subset of $\Omega_{\kappa h}$. For each $p \in \mathbb{R}$ and $h > 0$ introduce an operator H_h^p acting on functions v given on \mathbb{R}^d by the formula

$$H_h^p[v](x) = H(p, x, v(x), \delta_h v(x), \Delta_h v(x)).$$

Theorem 2.6. *Assume that for each $h \in (0, h_0]$ and each p we are given a bounded function v_h^p on \mathbb{R}^d which satisfies the equation*

$$H_h^p[v_h^p] = 0 \tag{2.8}$$

in D_h . Introduce

$$v_h = v_h^0$$

and assume that there is a constant $N_0 \in [0, \infty)$ such that for $h \in (0, h_0]$ and $x \in D_h$ we have

$$|v_h(x)|, \quad |\delta_{h,i} v_h(x)|, \quad (\rho(x) - \nu h) |\Delta_{h,i} v_h(x)| \leq N_0$$

for all i . Then for $h \in (0, h_0]$

$$|v_h^p(x) - v_h(x)| \leq N(|p| + \sup_{\partial_h D} |v_h^p - v_h|) \tag{2.9}$$

for any $x \in \mathbb{R}^d$ and $p \in \mathbb{R}$, where the constant N is independent of x, p , and h .

Proof. Take a constant N_1 to be specified later and introduce

$$w_h = v_h + N_1 p \Phi.$$

By Lemma 2.1 for each x

$$\begin{aligned} & H_h^p[w_h](x) - H[v_h](x) \\ &= N_1 p a_j \Delta_{h,i} \Phi(x) + N_1 p b_j \delta_{h,j} \Phi(x) + N_1 p c \Phi(x) + f, \end{aligned} \tag{2.10}$$

where a_j, b_j, c are some numbers satisfying (2.1) and f is such that

$$|f| \leq K|p|(1 + \sum_j (|\Delta_{h,j} v_h(x)| + |\delta_{h,j} v_h(x)|) + |v_h(x)|).$$

Observe that the second term on the left in (2.10) vanishes in D_h . Furthermore, $0 \leq (\rho - \nu h)^{-1} \leq 2\rho^{-1}$ if $\rho \geq 2\nu h$, so that, owing to the fact that $\kappa \geq 2\nu$ and $D_h \subset \Omega_{\kappa h}$, in D_h we obtain

$$|f| \leq N_2 |p| \rho^{-1},$$

where N_2 is independent of h and p . Finally, we set $N_1 = N_2$, take into account (2.7) and the fact that $D_h \subset \Omega_{\mu h}$ (since $\kappa \geq \mu$), and we conclude from (2.10) that

$$H_h^p[w_h] \leq 0 \tag{2.11}$$

in D_h if $p \geq 0$. Upon comparing this with (2.8) and using (2.3) we get that for any x

$$\begin{aligned} v_h^p(x) &\leq w_h(x) + \sup_{\partial_h D} (v_h^p - w_h)_+ \\ &\leq v_h(x) + Np + \sup_{\partial_h D} (v_h^p - v_h)_+, \end{aligned} \quad (2.12)$$

where N is independent of x, h and p .

If $p \leq 0$ the inequality in (2.11) is reversed and one gets

$$v_h - Np \leq w_h \leq v_h^p + \sup_{\partial_h D} (w_h - v_h^p)_+ \leq v_h^p + \sup_{\partial_h D} (v_h - v_h^p)_+.$$

By combining this with (2.12) we come to (2.9) and the theorem is proved.

Corollary 2.7. *Assume that $H(x, u)$ is locally Lipschitz continuous with respect to (x, u) and at all points of its differentiability with respect to (x, u)*

$$|H_{x_i}(x, u)| \leq K(1 + |u|)$$

for all i . Let $H(x, 0)$ be bounded on \mathbb{R}^d . For $h \in (0, \delta/K]$ denote by v_h a unique bounded solution of (2.4) in Ω_h with boundary condition $v = g$ on $\partial_h \Omega$ (see Theorem 2.2), where $g \in C^{0,1}(\mathbb{R}^d)$.

Finally, assume that there is a constant $N_0 \in [0, \infty)$ such that for $h \in (0, h_0]$ in Ω_h we have

$$|\delta_{h,i} v_h(x)|, \quad (\rho(x) - \nu h) |\Delta_{h,i} v_h(x)| \leq N_0$$

for all i .

Then for all $h \in (0, h_0]$ and $x, y \in \mathbb{R}^d$

$$|v_h(x) - v_h(y)| \leq N(|x - y| + h), \quad (2.13)$$

where the constant N is independent of x, y , and h .

Proof. If $|x - y| \geq h$ one can split the straight segment between x and y into adjacent pieces of length h combined with a remaining one of length less than h . This shows that we need only prove (2.13) for $|x - y| \leq h$.

Then fix a unit vector $l \in \mathbb{R}^d$ and for $p \in \mathbb{R}$ redefine H if necessary by setting

$$\begin{aligned} H(p, x, u) &= H(x + pl, u) \quad \text{if } |p| \leq h, \\ H(p, x, u) &= H(x + l \operatorname{sign} p, u) \quad \text{if } |p| \geq h, \end{aligned}$$

that is

$$H(p, x, u) = H(x + l\phi(p), u),$$

where $\phi(p) = (-h) \vee p \wedge h$. Observe that the function

$$v_h^p(x) := v_h(x + l\phi(p))$$

satisfies (2.8) (with new H) in $\Omega_{2h} \supset \Omega_{(\kappa+1)h}$, where the inclusion follows from the fact that $\kappa \geq \mu \geq 1$. By Lemma 2.5 there is a constant $N_1 \in (0, \infty)$ such that $|v_h - g| \leq N_1 \rho$ for $h \in (0, h_0]$.

Now set $D = \Omega_{\kappa h}$. Then $D_h = \Omega_{(\kappa+1)h}$ and we infer from Theorem 2.6 that for $|p| \leq h$

$$\begin{aligned} |v_h(x+lp) - v_h(x)| &\leq N(|p| + \sup_{\Omega_{(\kappa+1)h}^c} |v_h(\cdot + lp) - v_h|) \\ &\leq N(|p| + \sup_{\Omega_{(\kappa+2)h}^c} |v_h - g| + \sup_{\Omega_{(\kappa+1)h}^c} |v_h - g| + \sup_{\Omega_{(\kappa+1)h}^c} |g(\cdot + lp) - g|). \end{aligned} \quad (2.14)$$

Since $|v_h - g| \leq N_1\rho$, $|v_h - g| \leq N_1(\kappa+2)h$ outside $\Omega_{(\kappa+2)h}$, and this along with (2.14) and the arbitrariness of l proves (2.13) for $|x-y| \leq h \leq h_0$, which finishes proving the corollary.

3. TWO RESULTS FROM [16] AND [17]

We suppose that the assumptions in Section 1 are satisfied and take “cut-off” functions

$$\eta \in C_b^2(\mathbb{R}^d), \quad |\eta| \leq 1, \quad \zeta = \eta^2.$$

Fix an $h \in (0, \delta/(2K)]$ and set

$$\Lambda_{h,1} = h\Lambda, \quad \Lambda_{h,n+1} = \Lambda_{h,n} + h\Lambda, \quad n \geq 1, \quad \Lambda_{h,\infty} = \bigcup_n \Lambda_{h,n},$$

Define

$$\begin{aligned} Q^o &= \{x \in \Lambda_{h,\infty} : x + 3hB \subset \Omega\} = \Lambda_{h,\infty} \cap \Omega_{3h}, \\ Q &= \{x + \Lambda_{h,2} : x \in Q^o\}, \quad \delta Q = Q \setminus Q^o. \end{aligned}$$

Observe that $Q \subset \Omega_h$ and Q is a finite set. The latter is due to Assumption 1.2(iii) and follows from the fact that the number of points with integral coordinates lying in a bounded domain is finite combined with the fact that there is a number M such that the coordinates of all points in $M\Lambda_{1,\infty}$ are integers.

The following is a specification of Theorem 1.2 of [16] in the present setting.

Theorem 3.1. *Let v satisfy (1.5) in Q . Then there is a constant $N_1 \geq 1$ depending only on d_1, K , and δ such that, for any ν satisfying*

$$\nu \geq N_1(\sup_{\mathbb{R}^d} |D^2\eta| + \sup_{\mathbb{R}^d} |D\eta|^2 + 1),$$

we have

$$\max_{k,Q} \zeta |\delta_{h,k} v| \leq N_1(\sqrt{\nu} + \frac{1}{\nu}) \max_Q |v| + \frac{N_1}{\sqrt{\nu}} + \frac{N_1}{\nu} + N_1 \max_{k,\delta Q} \zeta |\delta_{h,k} v|. \quad (3.1)$$

Remark 3.1. In [16] a more general statement than Theorem 3.1 is proved under the assumption that $a^\alpha, b^\alpha, c^\alpha, f^\alpha$ are in $C^{0,1}(\mathbb{R}^d)$ rather than in $C^{1,1}(\mathbb{R}^d)$. Also δQ in (3.1) is replaced with a “thinner” set and Assumption 1.2(ii) is not used.

Now comes a version of Theorem 1.1 of [17].

Theorem 3.2. *Let v satisfy (1.5) in Q . Then there exists a constants N_2 depending only on K, d_1 , and δ such that if a constant $\nu > 0$ satisfies*

$$\nu \geq N_2(\sup_{\mathbb{R}^d} |D^2\eta| + \sup_{\mathbb{R}^d} |D\eta|^2 + 1),$$

then

$$\begin{aligned} \max_{Q,i,j} \zeta |\delta_{h,i} \delta_{h,j} v| &\leq \max_{\delta Q,i,j} \zeta |\delta_{h,i} \delta_{h,j} v| \\ &+ N_2(\nu^{1/2} + \sup_{\mathbb{R}^d} |D\eta|) \max_{Q,i} |\delta_{h,i} v| + N_2(1 + \max_Q |v|). \end{aligned}$$

4. PROOF OF THEOREM 1.2

(i) For $x \in \mathbb{R}^d$ and $u = (u', u'')$, where

$$u' = (u'_{-d_1}, \dots, u'_{-1}, u'_0, u'_1, \dots, u'_{d_1}), \quad u'' = (u''_{\pm 1}, \dots, u''_{\pm d_1}),$$

introduce

$$H(x, u) = \sup_{\alpha \in A} [a_k^\alpha(x) u_k'' + b_k^\alpha(x) u_k' - c^\alpha(x) u_0' + f^\alpha(x)].$$

By using the inequality

$$\sup_{\alpha} F^\alpha - \sup_{\alpha} G^\alpha \leq \sup_{\alpha} (F^\alpha - G^\alpha) \quad (4.1)$$

we easily conclude that $H(x, u)$ is locally Lipschitz continuous with respect to (x, u) . As such it is almost everywhere differentiable. Its derivatives are limits of finite differences and by using (4.1) again we see that at all points of differentiability of H we have

$$|H_{u_j'}|, |H_{u_i'}| \leq K, \quad j = \pm 1, \dots, \pm d_1, i = 0, \pm 1, \dots, \pm d_1, \quad H_{u_0'} \leq -\delta.$$

Furthermore,

$$\sup_{\alpha} F^\alpha - \sup_{\alpha} G^\alpha \geq \inf_{\alpha} (F^\alpha - G^\alpha),$$

which implies that $H_{u_j''} \geq \delta$.

We have checked that H (independent of p) satisfies Assumption 2.1. Now assertion (i) follows from Theorem 2.2 and the fact that $H(x, u)$ is a (Lipschitz) continuous in x , the latter being again a consequence of (4.1) and Assumption 1.1. This proves assertion (i) and combined with Lemma 2.5 shows that there is a constant $N \in (0, \infty)$ such that $|v_h - g| \leq N\rho$ for $h \in (0, h_0]$, which is part assertion (ii).

(ii) In light of (4.1) at all points of differentiability of $H(x, u)$ with respect to (x, u)

$$|H_{x_i}(x, u)| \leq N(K, d_1)(1 + |u|)$$

for all i . Hence, owing to Corollary 2.7 to prove assertion (ii), it suffices to prove (1.6), in which the first estimate is obtained above.

Note that if $x \in \Lambda_{h,\infty}$ and $x \notin Q^o$, then $\rho(x) \leq 3h$, and

$$|v_h(x + hl_k) - g(x + hl_k)|, |v_h(x) - g(x)| \leq Nh$$

for any k . Hence for $x \in \Lambda_{h,\infty} \setminus Q^o$ we have

$$\begin{aligned} |\delta_{h,k}v_h(x)| &\leq h^{-1}|v_h(x+hl_k) - v_h(x)| \\ &\leq h^{-1}(|v_h(x+hl_k) - g(x+hl_k)| + |v_h(x) - g(x)|) + N \leq N, \end{aligned}$$

where N is independent of h and x . In short

$$|\delta_{h,k}v_h| \leq N \quad (4.2)$$

on $\Lambda_{h,\infty} \setminus Q^o$, where N is independent of h .

By Theorem 3.1 with $\eta \equiv 1$ for all sufficiently small h

$$\max_{k,Q} |\delta_{h,k}v_h| \leq N(1 + \max_{\mathbb{R}^d} |v_h| + \max_{k,\delta Q} |\delta_{h,k}v_h|),$$

where N is independent of h . Since $|v_h - g| \leq N\rho$ and (4.2) holds on $\Lambda_{h,\infty} \setminus Q^o$, we conclude that (4.2) holds on $\Lambda_{h,\infty}$, provided that h is sufficiently small, where N is independent of h . Actually, (4.2) holds on \mathbb{R}^d just because any $x \in \mathbb{R}^d$ can be placed into an appropriate shift of $\Lambda_{h,\infty}$ with the shift not affecting any of the above constants N .

Thus, we estimated the second quantity in (1.6).

To estimate the last one we use Theorem 3.2. Once again it suffices to concentrate on points in $\Lambda_{h,\infty}$. Introduce $2\rho_0 = \max \rho$. It is a standard fact that for any $r \in (0, \rho_0]$ there exists an $\eta^{(r)} \in C_0^\infty(\Omega)$ such that

$$\eta^{(r)} = 1 \quad \text{on} \quad \Omega_{2r}, \quad \eta^{(r)} = 0 \quad \text{outside} \quad \Omega_r,$$

$$|\eta^{(r)}| \leq 1, \quad |D\eta^{(r)}| \leq N/r, \quad |D^2\eta^{(r)}| \leq N/r^2,$$

where the constant N is independent of r . Furthermore, since $Q^o = \Lambda_{h,\infty} \cap \Omega_{3h}$ and $Q \subset \Omega_h$ it holds that $\delta Q \subset \Omega \setminus \Omega_{3h}$. By taking $r \geq 3h$, $r \leq 1$, and applying Theorem 3.2 with $\eta = \eta^{(r)}$ (and $\nu = N/r^2$, where N is an appropriate constant independent of r and h), for sufficiently small $h > 0$ and any i, j we obtain on $\Lambda_{h,\infty}$ that

$$(\eta^{(r)}(x))^2 |\delta_{h,i}\delta_{h,j}v_h(x)| \leq N/r, \quad (4.3)$$

where N is independent of x, r , and h .

Now we use the arbitrariness of x and r . Take an $x \in \Lambda_{h,\infty}$ such that $\rho(x) \geq 6h$ and take $r = \rho(x)/2 (\geq 3h)$. Then $\eta^{(r)}(x) = 1$ and (4.3) yields $\rho(x)|\delta_{h,i}\delta_{h,j}v_h(x)| \leq N$ implying that

$$(\rho(x) - 6h)|\delta_{h,i}\delta_{h,j}v_h(x)| \leq N \quad (4.4)$$

whenever $x \in \Lambda_{h,\infty}$ and $\rho(x) \geq 6h$. However, (4.4) is obvious if $\rho(x) \leq 6h$. Thus (4.4) holds on $\Lambda_{h,\infty}$ and, as it was mentioned, on \mathbb{R}^d . The theorem is proved.

5. PROOF OF THEOREM 1.4

We need two additional auxiliary results. Estimate (5.3) is quite similar to (4.6) of [6] the latter being a particular case of the former which occurs when $N_1^* = 0$ (in (4.6) of [6] it is assumed that $u \in C^{0,1}$).

Lemma 5.1. *Let vector $l \in B$ be on the first basis axis in \mathbb{R}^d . Let $\varepsilon, h > 0$, $\eta \in C_0^\infty(\mathbb{R}^d)$ be a spherically symmetric function with support in $B_\varepsilon = \{x : |x| < \varepsilon\}$ and let u be a bounded Borel function on \mathbb{R}^d . Define*

$$B'_\varepsilon = \{z \in \mathbb{R}^{d-1} : |z| < \varepsilon\}.$$

Assume that for all $x, y \in [-2h - \varepsilon, 2h + \varepsilon] \times B'_\varepsilon$ we have

$$|u(x) - u(y)| \leq N_1|x - y| + N_1^*h, \quad |\Delta_{h,l}u(x)| \leq N_2. \quad (5.1)$$

Introduce $w = u * \eta$. Then

$$|D_l^2 w(0) - \Delta_{h,l} w(0)| \leq N_2 h^2 \|D_l^2 \eta\|_{L_1} + h^4 (N_1 \varepsilon + N_1^* h) \|D_l^6 \eta\|_{L_1}. \quad (5.2)$$

Also for $x = \gamma l$, $|x| \leq h$, we have

$$|D_l^2 w(x)| \leq N_2 \|\eta\|_{L_1} + h^2 (N_1 \varepsilon + N_1^* h) \|D_l^4 \eta\|_{L_1}, \quad (5.3)$$

Finally,

$$|D_l w(0) - \delta_{h,l} w(0)| \leq N_2 h \|\eta\|_{L_1} + h^3 (N_1 \varepsilon + N_1^* h) \|D_l^4 \eta\|_{L_1}. \quad (5.4)$$

Proof. First for $n = 1, 2, \dots, 6$ and $x = \gamma l$, $|x| \leq 2h$, we have

$$D_l^n w(x) = \int_{B_\varepsilon} u(x - y) D_l^n \eta(y) dy = \int_{B_\varepsilon} [u(x - y) - u(x)] D_l^n \eta(y) dy.$$

Hence,

$$|D_l^n w(x)| \leq (N_1 \varepsilon + N_1^* h) \|D_l^n \eta\|_{L_1}. \quad (5.5)$$

Next observe that for smooth functions $f(t)$ of one real variable t we have

$$f''(0) = \frac{f(h) - 2f(0) + f(-h)}{h^2} - \frac{h^2}{6} \int_{-1}^1 (1 - |t|)^3 f^{(4)}(th) dt.$$

Hence

$$|D_l^2 w(0) - \Delta_{h,l} w(0)| \leq h^2 \sup_{|t_1| \leq 1} |D_l^4 w(lt_1 h)|. \quad (5.6)$$

Therefore, to prove (5.2) it suffices to show that for $|t_1| \leq 1$

$$|D_l^4 w(lt_1 h)| \leq N_2 \|D_l^2 \eta\|_{L_1} + h^2 (N_1 \varepsilon + N_1^* h) \|D_l^6 \eta\|_{L_1}. \quad (5.7)$$

To this end we use (5.6) which implies that

$$|D_l^2 u * \eta(0)| \leq |\Delta_{h,l} u * \eta(0)| + h^2 \sup_{|t_1| \leq 1} |D_l^2 u * D_l^2 \eta(lt_1 h)|. \quad (5.8)$$

By applying (5.8) to $D_l^2 \eta$ in place of η and $lt_1 h$ in place of 0 we find

$$\begin{aligned} |D_l^4 w(lt_1 h)| &= |D_l^2 u * D_l^2 \eta(lt_1 h)| \leq |\Delta_{h,l} u * D_l^2 \eta(lt_1 h)| \\ &\quad + h^2 \sup_{|t_2| \leq 1} |D_l^2 u * D_l^4 \eta(l(t_1 + t_2)h)|, \end{aligned}$$

Here, owing to (5.1) and the fact that $|t_1 + t_2|h + \varepsilon \leq 2h + \varepsilon$,

$$|\Delta_{h,l}u * D_l^2 \eta(l(t_1 + t_2)h)| \leq N_2 \|D_l^2 \eta\|_{L_1}.$$

Furthermore, by (5.5) and the fact that $|t_1 + t_2|h \leq 2h$

$$\begin{aligned} |D_l^2 u * D_l^4 \eta(l(t_1 + t_2)h)| &= |u * D_l^6 \eta(l(t_1 + t_2)h)| \\ &\leq (N_1 \varepsilon + N_1^* h) \|D_l^6 \eta\|_{L_1}. \end{aligned}$$

This proves (5.7) and (5.2).

Since, as above, $|\Delta_{h,l}w(0)| \leq N_2 \|\eta\|_{L_1}$, we obtain (5.3) for $x = 0$ directly from (5.6) and (5.5). For other values of x we obtain (5.3) upon observing that the above argument is valid if we replace 0 and lt_1h in (5.6) with x and $x + lt_1h$, respectively.

To prove (5.4) observe that in the one-dimensional case

$$f'(0) = \frac{f(h) - f(0)}{h} - h \int_0^1 (1-t) f''(th) dt.$$

It follows that

$$|D_l w(0) - \delta_{h,l} w(0)| \leq h \int_0^1 (1-t) |D_l^2 w(lth)| dt \leq h \sup_{t \in [0,1]} |D_l^2 w(lth)|, \quad (5.9)$$

which along with (5.3) yields (5.4). The lemma is proved.

Remark 5.1. Take a nonnegative spherically symmetric $\zeta \in C_0^\infty(\mathbb{R}^d)$ whose integral is one and whose support is in B and for $\varepsilon > 0$ and locally integrable functions u on \mathbb{R}^d use the notation

$$u^{(\varepsilon)}(x) = \int_{\mathbb{R}^d} u(x - \varepsilon y) \zeta(y) dy.$$

Denote by v_h a unique bounded solution of (1.5), with zero boundary data on $\partial_h \Omega$. By Theorem 1.2 the function v_h is well defined at least for sufficiently small $h > 0$.

We want to explain why we iterated (5.8) and why only once. We will apply Lemma 5.1 to $u = v_h$. For simplicity assume that $b \equiv 0$ and notice that on Ω_h for any $\alpha \in A$ we have

$$a_k^\alpha D_{l_k}^2 v - c^\alpha v + f^\alpha \leq 0. \quad (5.10)$$

Next assume that a and c are independent of x as in [6]. Then we can mollify all terms in (5.10) and obtain that in $\Omega_{h+\varepsilon}$ we have

$$a_k^\alpha \Delta_{h,k} v_h^{(\varepsilon)} - c^\alpha v_h^{(\varepsilon)} + [f^\alpha]^{(\varepsilon)} \leq 0. \quad (5.11)$$

It is well known that $|[f^\alpha]^{(\varepsilon)} - f^\alpha| \leq N\varepsilon^2$ since $f^\alpha \in C^{1,1}$. Therefore, (5.11) implies that

$$a_k^\alpha \Delta_{h,k} v_h^{(\varepsilon)} - c^\alpha v_h^{(\varepsilon)} + f^\alpha \leq N\varepsilon^2.$$

Now we replace $\Delta_{h,k} v_h^{(\varepsilon)}$ with $D_{l_k}^2 v_h^{(\varepsilon)}$. Ignoring for a moment that our estimates of $\Delta_{h,k} v_h$ are local in Ω , we obtain from Lemma 5.1 that

$$a_k^\alpha D_{l_k}^2 v_h^{(\varepsilon)} - c^\alpha v_h^{(\varepsilon)} + f^\alpha \leq N[\varepsilon^2 + h^2 \varepsilon^{-2} + h^4 (\varepsilon + h) \varepsilon^{-6}] =: NM(h, \varepsilon).$$

It follows from Lemma 2.4, that for, perhaps, another constant N

$$H[v_h^{(\varepsilon)} + N\Psi M(h, \varepsilon)] \leq 0$$

in $\Omega_{h+\varepsilon}$ By the maximum principle

$$v \leq v_h^{(\varepsilon)} + NM(h, \varepsilon) + \sup_{\Omega_{h+\varepsilon}^c} (v - v_h^{(\varepsilon)})_+. \quad (5.12)$$

It seems that the best possible estimate of the last term is that it is less than $N(h + \varepsilon)$ and we will prove this estimate later. Then

$$v \leq v_h^{(\varepsilon)} + N[M(h, \varepsilon) + h + \varepsilon],$$

$$v \leq v_h + N[M(h, \varepsilon) + h + \varepsilon] + \sup(v_h^{(\varepsilon)} - v_h)_+. \quad (5.13)$$

Let us ignore the contribution of the last term in the right-hand side and try to make $M(h, \varepsilon) + h + \varepsilon$ as small as possible on the account of arbitrariness in choosing ε . Observe that this quantity contains $\varepsilon + h^2\varepsilon^{-2}$ which is bigger than $\gamma h^{2/3}$, where the constant $\gamma > 0$ is independent of h . Furthermore, $\varepsilon + h^2\varepsilon^{-2} = 2h^{2/3}$ when $\varepsilon = h^{2/3}$. With this ε

$$h^4(\varepsilon + h)\varepsilon^{-6} = O(h^{2/3})$$

as well and we obtain that $v \leq v_h + Nh^{2/3}$ for sufficiently small $h > 0$.

This is roughly the way we are going to use (5.2).

Now imagine that we did not enhance the estimate of $D_l^4 w$ and after (5.6) just used (5.5) to obtain

$$|D_l^2 v_h^{(\varepsilon)} - \Delta_{h,l} v_h^{(\varepsilon)}| \leq Nh^2(\varepsilon + h)\varepsilon^{-4}.$$

Then we would obtain (5.13) with

$$M(h, \varepsilon) = \varepsilon^2 + h^2(\varepsilon + h)\varepsilon^{-4}$$

and $M(h, \varepsilon) + h + \varepsilon$ would contain the term $\varepsilon + h^2\varepsilon^{-3}$ whose minimum with respect to ε is of order $h^{1/2}$ and is comparable with its value at $\varepsilon = h^{1/2}$. Then at best we would have that $v \leq v_h + Nh^{1/2}$.

On the other hand, we could iterate (5.8) one more time and obtain that

$$\begin{aligned} |D_l^2 u * D_l^4 \eta(l(t_1 + t_2)h)| &\leq |\Delta_{h,l} u * D_l^4 \eta(l(t_1 + t_2)h)| \\ &+ h^2 \sup_{|t_3| \leq 1} |D_l^2 u * D_l^6 \eta(l(t_1 + t_2 + t_3)h)|, \end{aligned}$$

which in our case leads to

$$\begin{aligned} |D_l^2 v_h^{(\varepsilon)} - \Delta_{h,l} v_h^{(\varepsilon)}| &\leq Nh^2[\varepsilon^{-2} + h^2(\varepsilon^{-4} + h^2(\varepsilon + h)\varepsilon^{-8})] \\ &= N(h^2\varepsilon^{-2} + h^4\varepsilon^{-4} + h^6(\varepsilon + h)\varepsilon^{-8}). \end{aligned}$$

This time again $M(h, \varepsilon) + h + \varepsilon$ contains $\varepsilon + h^2\varepsilon^{-2}$, which will not lead to a better rate than $h^{2/3}$.

It is seen from the above that the fact that $|[f^\alpha]^{(\varepsilon)} - f^\alpha| \leq N\varepsilon^2$ plays no significant role. The estimate $|[f^\alpha]^{(\varepsilon)} - f^\alpha| \leq N\varepsilon$ would do equally well. Also in this framework we will be satisfied with estimating the last term in the right-hand side of (5.13) just by $N\varepsilon$, which is quite easy.

It is worth noting that the situation with constant in x coefficients a, b , and c is quite different in the whole space (see [6]). There no boundary term like the last term in (5.12) appears and one ends up with $v \leq v_h^{(\varepsilon)} + NM(h, \varepsilon)$. With some additional effort (even in our setting of bounded smooth Ω and variable coefficients, see Lemma 5.3) one can prove that $|v_h^{(\varepsilon)} - v_h| \leq N(\varepsilon^2 + h)$. Then

$$v \leq v_h + N[h + \varepsilon^2 + h^2\varepsilon^{-2} + h^4(\varepsilon + h)\varepsilon^{-6}]. \quad (5.14)$$

The minimum of $\varepsilon^2 + h^2\varepsilon^{-2}$ with respect to ε is proportional to its value at $\varepsilon = h^{1/2}$ and is of order h . Other error terms on the right in (5.14) are of the same or higher order. In this way it is proved in [6] that $v \leq v_h + Nh$. It is also shown there that in general this estimate is optimal, so that there is no need to even consider additional iterations of (5.8).

Finally, we point out that even for the equations in the whole space with *variable* coefficients the error term of order ε still appears in the transition from (5.10) to (5.11). The method of “shaking the coefficients” produces an error of the same order, which allows us not to use this method on the account that we have a good control of the second-order differences of v_h .

Lemma 5.2. *There exists a constant N such that for all sufficiently small $h > 0$, for any $\varepsilon > 0$,*

- (i) *In \mathbb{R}^d we have $|v_h^{(\varepsilon)} - v_h| \leq N(\varepsilon + h)$;*
- (ii) *In $\Omega_{4\varepsilon+16h}$ for any $\alpha \in A$ we have*

$$\begin{aligned} & a_k^\alpha D_{l_k}^2 v_h^{(\varepsilon)} + b_k^\alpha D_{l_k} v_h^{(\varepsilon)} - c^\alpha v_h^{(\varepsilon)}(x) + f^\alpha \\ & \leq N(h + \varepsilon + h^2\varepsilon^{-2})\rho^{-1} + Nh^3\varepsilon^{-6}(\varepsilon + h)(\varepsilon^2 + h). \end{aligned} \quad (5.15)$$

Proof. Assertion (i) follows immediately from the fact that $|v_h(x) - v_h(y)| \leq N(|x - y| + h)$ (see Theorem 1.2).

- (ii) Fix an $\alpha \in A$ and observe that for $x \in \Omega_{h+\varepsilon}$ we have

$$[a_k^\alpha \Delta_{h,k} v_h + b_k^\alpha \delta_{h,k} v_h - c^\alpha v_h + f^\alpha]^{(\varepsilon)}(x) \leq 0. \quad (5.16)$$

Next,

$$\begin{aligned} & [a_k^\alpha \Delta_{h,k} v_h]^{(\varepsilon)}(x) = a_k^\alpha(x) \Delta_{h,k} v_h^{(\varepsilon)}(x) \\ & + \int_B [a_k^\alpha(x + \varepsilon y) - a_k^\alpha(x)] \zeta(y) \Delta_{h,k} v_h(x + \varepsilon y) dy, \end{aligned}$$

where owing to Theorem 1.2, for $x \in \Omega_{6h+\varepsilon}$ the last term by magnitude is less than

$$N\varepsilon \int_B \frac{1}{\rho(x + \varepsilon y) - 6h} dy,$$

which is less than $N\varepsilon\rho^{-1}(x)$ if $x \in \Omega_{12h+2\varepsilon}$ since then

$$\begin{aligned} \rho(x + \varepsilon y) - \rho(x) & \geq -\varepsilon, \quad \rho(x + \varepsilon y) - (1/2)\rho(x) \geq (1/2)\rho(x) - \varepsilon \geq 6h, \\ \rho(x + \varepsilon y) - 6h & \geq (1/2)\rho(x). \end{aligned}$$

Hence in $\Omega_{12h+2\varepsilon}$

$$a_k^\alpha \Delta_{h,k} v_h^{(\varepsilon)} \leq [a_k^\alpha \Delta_{h,k} v_h]^{(\varepsilon)} + N\varepsilon \rho^{-1}.$$

Then again owing to Theorem 1.2 in \mathbb{R}^d

$$\begin{aligned} b_k^\alpha \delta_{h,k} v_h^{(\varepsilon)} &\leq [b_k^\alpha \delta_{h,k} v_h]^{(\varepsilon)} + N\varepsilon, \quad -c^\alpha v_h^{(\varepsilon)} \leq -[c^\alpha v_h]^{(\varepsilon)} + N\varepsilon, \\ f^\alpha &\leq [f^\alpha]^{(\varepsilon)} + N\varepsilon. \end{aligned}$$

Coming back to (5.16) we conclude that in $\Omega_{12h+2\varepsilon}$ we have

$$a_k^\alpha \Delta_{h,k} v_h^{(\varepsilon)} + b_k^\alpha \delta_{h,k} v_h^{(\varepsilon)} - c^\alpha v_h^{(\varepsilon)} + f^\alpha(x) \leq N\varepsilon \rho^{-1}. \quad (5.17)$$

Now we are going to use Lemma 5.1 in order to replace $\Delta_{h,k}$ and $\delta_{h,k}$ in (5.17) with $D_{l_k}^2$ and D_{l_k} , respectively. Of course, this time we take $\eta(x) = \varepsilon^{-d} \zeta(x/\varepsilon)$ and $u = v_h$ in Lemma 5.1. This lemma is stated only for vectors on the first basis axis. The reader understands that one can prove an appropriately modified statements for any vector $l \in B$ under the assumption that (5.1) holds in the cylinder $C_{2\varepsilon+4h,\varepsilon,l}$ centered at the origin with height $2\varepsilon + 4h$, base radius ε , and axis parallel to l . Naturally, one can move such cylinders to be centered at any point x_0 . Observe that if $x_0 \in \Omega_{2\varepsilon+8h}$, then, for any $l \in \Lambda$, all points of the cylinder $x_0 + C_{2\varepsilon+4h,\varepsilon,l}$ are at a distance not less than

$$\rho(x_0) - 2\varepsilon - 2h > 6h$$

from Ω^c . Hence, while applying Lemma 5.1 to $x_0 + C_{2\varepsilon+4h,\varepsilon,l}$ we can take the constant N_2 to be

$$N(\rho(x_0) - 2\varepsilon - 8h)^{-1},$$

where N is the constant from Theorem 1.2. Notice that

$$\rho(x_0) - 2\varepsilon - 8h \geq 1/2 \rho(x_0)$$

if $x_0 \in \Omega_{4\varepsilon+16h}$. Therefore, on $\Omega_{4\varepsilon+16h}$ by Lemma 5.1 we obtain

$$\begin{aligned} |D_{l_k} v_h^{(\varepsilon)} - \delta_{h,k} v_h^{(\varepsilon)}| &\leq Nh\rho^{-1} + Nh^3(\varepsilon + h)\varepsilon^{-4}, \\ |D_{l_k}^2 v_h^{(\varepsilon)} - \Delta_{h,k} v_h^{(\varepsilon)}| &\leq Nh^2\varepsilon^{-2}\rho^{-1} + Nh^4(\varepsilon + h)\varepsilon^{-6}. \end{aligned}$$

These estimates and (5.17) yield (5.15). The lemma is proved.

Assertion (i) of Lemma 5.2 that $|v_h^{(\varepsilon)} - v_h| \leq N(\varepsilon + h)$ in \mathbb{R}^d can be improved for points at a fixed distance from Ω^c if one uses the following lemma, in which e_1, \dots, e_d is the standard orthonormal basis in \mathbb{R}^d .

Lemma 5.3. *Let u be a bounded Borel function on \mathbb{R}^d . Assume that for all $x, y \in [-(2h + \varepsilon), 2h + \varepsilon]^d$ we have*

$$\begin{aligned} |u(x) - u(y)| &\leq N_1(|x - y| + h), \\ |\Delta_{h,e_i} u(x)| &\leq N_2 \quad i = 1, \dots, d. \end{aligned}$$

Then for $\varepsilon \geq h$

$$|u^{(\varepsilon)}(0) - u(0)| \leq N(\varepsilon^2 + h), \quad (5.18)$$

where the constant N depends only on ζ, d, N_1 , and N_2 .

Proof. We follow closely the argument in Remark 4.5 of [6]. For $r > 0$ introduce

$$w_r(x) = r^{-d/2} \int_{\mathbb{R}^d} \zeta_r(x-y) u(y) dy,$$

where

$$\zeta_r(x) = \zeta(xr^{-1/2}).$$

Simple computations show that

$$\frac{\partial}{\partial r} \left[\frac{1}{r^{d/2}} \zeta_r(x) \right] = \frac{1}{r^{d/2}} D_i^2 \zeta_{i,r}(x),$$

where (with no summation in i)

$$\zeta_{i,r}(x) = \zeta_i(xr^{-1/2}), \quad \zeta_i(x) = -\frac{1}{2} \int_{-\infty}^{x_i} \zeta(x - x_i e_i + s e_i) s ds.$$

Observe that since ζ is spherically symmetric, the support of ζ_i is in B and, of course, $\zeta_i \in C_0^\infty(\mathbb{R}^d)$. Also notice for the future that (no summation in i)

$$|D_i^4 \zeta_{i,r}(x)| = \frac{1}{r^2} |D_i^4 \zeta_i|(xr^{-1/2}) \leq \frac{1}{r^2} (|D_i^3 \zeta| + |D_i^2 \zeta|)(xr^{-1/2}).$$

It follows that

$$w_t(0) - w_r(0) = \int_r^t \frac{1}{s^{d/2}} D_i^2 [u * \zeta_{i,s}](0) ds.$$

The support of $\zeta_{i,s}$ lies in $B_{\sqrt{s}}$. Therefore, if $0 < r < t \leq \varepsilon^2$ and $s \in [r, t]$, we can use (5.3) and obtain (no summation in i)

$$\begin{aligned} |D_i^2 [u * \zeta_{i,s}](0)| &\leq N_2 \|\zeta_{i,s}\|_{L_1} + N_1 h^2 (s^{1/2} + h) \|D^4 \zeta_{i,s}\|_{L_1} \\ &= N_2 s^{d/2} \|\zeta_i\|_{L_1} + N_1 h^2 (s^{1/2} + h) s^{d/2-2} \|D^4 \zeta_i\|_{L_1}. \end{aligned}$$

Hence

$$\begin{aligned} |w_t(0) - w_r(0)| &\leq N \int_r^t (1 + h^2 s^{-3/2} + h^3 s^{-2}) ds \\ &\leq N(t-r) + N h^2 r^{-1/2} + N h^3 r^{-1}, \end{aligned}$$

where and below the constants N depend only on ζ, d, N_1 , and N_2 . We combine this with

$$|w_r(0) - u(0)| = \left| \int_{\mathbb{R}^d} \zeta(y) [u(yr^{1/2}) - u(0)] dy \right| \leq N_1 (r^{1/2} + h).$$

Then we get

$$|w_t(0) - u(0)| \leq N(t-r) + N h^2 r^{-1/2} + N h^3 r^{-1} + N(r^{1/2} + h),$$

which leads to (5.18) if we take $t = \varepsilon^2$ and $r = h^2$. The lemma is proved.

Proof of Theorem 1.4. Take a constant $\mu \geq 1$ from Lemma 2.4. By Lemmas 5.2(ii) and 2.4, there is a constant N independent of α, h , and ε such that for

$$N_1 := N(h + \varepsilon + h^2 \varepsilon^{-2}) + N h^3 \varepsilon^{-6} (\varepsilon + h)(\varepsilon^2 + h),$$

$$w_h^\varepsilon := v_h^{(\varepsilon)} + N_1 \Phi$$

and sufficiently small $h > 0$ we have

$$a_k^\alpha D_{l_k}^2 w_h^\varepsilon + b_k^\alpha D_{l_k} w_h^\varepsilon - c^\alpha w_h^\varepsilon + f^\alpha \leq 0$$

for all $\alpha \in A$ in Ω_κ , where $\kappa = (\mu h) \vee (4\varepsilon + 16h)$. By the maximum principle

$$v \leq w_h^\varepsilon + \max_{\Omega \setminus \Omega_\kappa} (v - w_h^\varepsilon)_+ \leq v_h^{(\varepsilon)} + N_1 \Phi + \max_{\Omega \setminus \Omega_\kappa} (v - v_h^{(\varepsilon)})_+ \quad (5.19)$$

in Ω . By Theorem 1.3 we have $|v - g| \leq N\kappa$ in $\Omega \setminus \Omega_\kappa$. Furthermore, by Lemma 5.2(i) we have $|v_h^{(\varepsilon)} - v_h| \leq N(\varepsilon + h)$ everywhere which implies that

$$|v_h^{(\varepsilon)} - g| \leq |v_h - g| + N(\varepsilon + h)$$

and along with Theorem 1.2 yields that

$$|v_h^{(\varepsilon)} - g| \leq N\kappa + N(\varepsilon + h)$$

in $\Omega \setminus \Omega_\kappa$. Upon combining this with (5.19) and observing that $\kappa \leq N(\varepsilon + h)$ we get

$$v \leq v_h + N(h + \varepsilon + h^2 \varepsilon^{-2}) + Nh^3 \varepsilon^{-6}(\varepsilon + h)(\varepsilon^2 + h),$$

which yields that in Ω for sufficiently small $h > 0$ we have

$$v \leq v_h + Nh^{2/3} \quad (5.20)$$

if we set $\varepsilon = h^{2/3}$.

To prove that

$$v_h \leq v + Nh^{2/3} \quad (5.21)$$

we reverse the roles of v_h and v . In Ω_ε we have

$$[a_k^\alpha D_{l_k}^2 v + b_k^\alpha D_{l_k} v - c^\alpha v + f^\alpha]^{(\varepsilon)} \leq 0. \quad (5.22)$$

Furthermore, for $x \in \Omega_\varepsilon$

$$a_k^\alpha D_{l_k}^2 v^{(\varepsilon)}(x) \leq [a_k^\alpha D_{l_k}^2 v]^{(\varepsilon)} + N\varepsilon \sup_{k, x+\varepsilon B} |D_{l_k}^2 v|,$$

where by Theorem 1.3 the second term on the right is dominated by $N(\rho(x) - \varepsilon)^{-1}$, which is less than $N\rho^{-1}(x)$ if $\rho(x) - \varepsilon \geq (1/2)\rho(x)$, that is if $x \in \Omega_{2\varepsilon}$. Similarly one estimates $b_k^\alpha D_{l_k} v^{(\varepsilon)}$, $-c^\alpha v^{(\varepsilon)}$ and $[f^\alpha]^{(\varepsilon)}$. Then one concludes from (5.22) that

$$a_k^\alpha D_{l_k}^2 v^{(\varepsilon)} + b_k^\alpha D_{l_k} v^{(\varepsilon)} - c^\alpha v^{(\varepsilon)} + f^\alpha \leq N\varepsilon \rho^{-1} \quad (5.23)$$

in $\Omega_{2\varepsilon}$.

Next, as in (5.6) for $x \in \Omega_{2h+2\varepsilon}$ we have

$$|D_{l_k}^2 v^{(\varepsilon)}(x) - \Delta_{h,k} v^{(\varepsilon)}(x)| \leq Nh^2 \sup_{x+hB} |D_{l_k}^4 v^{(\varepsilon)}(y)|,$$

where

$$|D_{l_k}^4 v^{(\varepsilon)}(y)| = |D_{l_k}^2 [D_{l_k}^2 v]^{(\varepsilon)}(y)| \leq N\varepsilon^{-2}(\rho(x) - h - \varepsilon)^{-1} \leq N\varepsilon^{-2}\rho^{-1}(x)$$

if $y \in x + hB$. Also as in (5.9)

$$|D_{l_k} v^{(\varepsilon)}(x) - \delta_{h,k} v^{(\varepsilon)}(x)| \leq Nh \sup_{x+hB} |D_{l_k}^2 v^{(\varepsilon)}|,$$

where

$$|D_{l_k}^2 v^{(\varepsilon)}(y)| \leq N(\rho(x) - h - \varepsilon)^{-1} \leq N\rho^{-1}(x)$$

if $y \in x + hB$ and $x \in \Omega_{2h+2\varepsilon}$. Hence (5.23) implies that

$$a_k^\alpha \Delta_{h,k} v^{(\varepsilon)} + b_k^\alpha \delta_{h,k} v^{(\varepsilon)} - c^\alpha v^{(\varepsilon)} + f^\alpha \leq N(\varepsilon + h^2 \varepsilon^{-2} + h) \rho^{-1}$$

in $\Omega_{2h+2\varepsilon}$. At this point it is convenient to extend v outside Ω as g . By using Lemma 2.4 and the maximum principle as above we obtain that in \mathbb{R}^d

$$v_h \leq v^{(\varepsilon)} + N(\varepsilon + h^2 \varepsilon^{-2} + h) + \sup_{\partial_\chi \Omega} (v_h - v^{(\varepsilon)})_+,$$

where $\chi = (\mu h) \vee (2h + 2\varepsilon)$. Furthermore, in $\Omega \setminus \Omega_\chi$ we have $|v_h - g| \leq N\chi$, which follows from Theorem 1.2(ii) and $|v^{(\varepsilon)} - g| \leq N(\chi + \varepsilon) \leq N\chi$, which follows from the fact that $v \in C^{0,1}(\bar{\Omega})$ by Theorem 1.3. Since $\chi \leq N(h + \varepsilon)$ we conclude that

$$v_h \leq v^{(\varepsilon)} + N(\varepsilon + h^2 \varepsilon^{-2} + h).$$

The boundedness of the first derivatives of v implies that $v^{(\varepsilon)} \leq v + N\varepsilon$. Hence,

$$v_h \leq v + N(\varepsilon + h^2 \varepsilon^{-2} + h),$$

which yields (5.21) for $\varepsilon = h^{2/3}$ and along with (5.20) brings the proof of the theorem to an end.

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